Quark-Antiquark Potential and Generalized Borel Transform *

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Abstract

The heavy quark potential and particularly the one proposed by Richardson to incorporate both asymptotic freedom and linear confinement is analyzed in terms of a generalized Borel transform recently proposed. We were able to obtain the potential behaviour in the coordinate space, including intermediate distances in the range of physical interest. The analytic expression for the potential gives rise to uncertainties much smaller than Λ of QCD.

Classification: 12.38 Aw - 12.38 Lg
Keywords: quark-antiquak potential; Borel transform; analytic properties.

*Partially supported by CONICET and ANPCyT-Argentina.

Among the different proposals for describing quark-antiquark interactions, the Richardson potential [1], due to its is simple structure, has been the subject of continuous interest [2]. This potential requieres, in the author words, the minimal number of parameters. In fact, the only parameter entering the potential is the QCD related scale Λ. This potential, designed in order to present both asymptotic freedom and linear quark confinement includes the single dressed gluon exchange amplitude, namely

$$V(q) = \frac{4}{3}\,\frac{\alpha_s(q^2)}{q^2} \tag{1}$$

Asymptotic freedom is present as soon as one adopts for $\alpha_s(q^2)$ the effective running coupling constant provided by the renormalization group. Quark confinement is imposed by requiring that for q small $V(q)$ behaves as the inverse four power of q that guarantees a linear behaviour in r. Then, the Richardson potential reads

$$V^R(q) = \frac{16\,\pi^2\,C_F}{\beta_0}\,\frac{1}{q^2\,\ln(1+q^2/\Lambda^2)} \tag{2}$$

where C_F is a group coefficient.

It is clear that the explicit calculation of the QCD coupling constant in position space is crucial when the Richardson potential is to be applied in a concrete case. This is because the Fourier transform of (2) provides the configuration space expression

$$\overline{V}^R(r) = -\frac{C_F}{r}\,\frac{2\,\pi}{\beta_0}\,\overline{\alpha}(1/r) = -\frac{C_F}{r}\,\frac{2\,\pi}{\beta_0}\,\left[a(1/r) - \Lambda^2\,r^2\right] \tag{3}$$

Here

$$a(1/r) = 1 - 4\,f(r) \tag{4}$$

$$f(r) \equiv \int_1^\infty \frac{dq}{q}\,\frac{e^{-q\,\Lambda\,r}}{\left[\ln(q^2-1)\right]^2+\pi^2} \tag{5}$$

This expression was only computed numerically. There exist some analytical results corresponding to some asymptotic conditions. For example, for $\Lambda\,r \ll 1$, the Richardson potential was shown to behave softly than the Coulomb interaction [3], namely

$$\overline{V}^R(r) \to -\frac{1}{\Lambda\,r\,\ln(\Lambda\,r)} \tag{6}$$

and for $\Lambda\,r \gg 1$ provides linear confinement.

Our main point in this paper is the calculation of the strong coupling constant in position space starting from the last integral representation eq.(5). In so doing we provide either an input for the Richardson potential in configuration space or to any other alternative proposal for the quark potential including the original QCD running coupling constant or any alternative expression [4]. To this end we fully analyze the analytic structure of the integral in the Borel plane [5]. In so doing we are able to obtain the potential behaviour as a function of r, including intermediate distances.

Any perturbative analysis starts from the general relationship between $V(q)$ and $\bar{V}(r)$ that ends with the corresponding relation between the couplings $\alpha_s(q)$ and $\bar{\alpha}(1/r)$. Notice that in the perturbative calculation, $\overline{\alpha}(1/r)$ coincides with $a(1/r)$. Moreover, it it has been shown [6] that one can write, for any static potential,

$$\overline{\alpha}(1/r) = \sum_n f_n \left[-\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right]^n \alpha_s(q = \kappa/r) \tag{7}$$

where f_n are known constants, $\kappa = exp(\gamma_E)$ a constant and $\beta(\alpha_s) \equiv \mu^2 \, \partial\alpha_s(q)/\partial\mu^2$. In the case of the Richardson potential, this series is asymptotically ($q^2 \gg 1$) factorial divergent and its Borel sum does not exist, namely

$$\overline{\alpha}(1/r) \sim \alpha_s(q = \kappa/r) \sum_n f_n \left[\beta_0 \, \alpha_s(q = \kappa/r) \right]^n n!$$

Certainly, this expression provides sensible results only for very small distances because at increasing distances the non-perturbative contributions start to be important.

It is worth to mention that the analytic behaviour of $\bar{\alpha}(1/r)$ has been studied [7] by summing the divergent asymptotic series by using the standard Borel formalism. Clearly, being the expression No Borel "sumable", the Borel transform $B(s)$ has singularities for different values s_k on the integration path of

$$\overline{\alpha}(1/r) = \int_0^\infty exp \left[-s/\alpha(q = \kappa/r) \right] \, B(s) \, ds$$

Consequently, it can be defined only in principal value, showing ambiguities coming from the exponential in this integral. This approach implies that the non-perturbative contribution is considered of the same order of magnitude as the ambiguity inherent to the method [8]. An additional problem coming from the use of a perturbative α_s is the presence of the Landau pole, conditioning the validity of any amplitude representing any physical observable to a finite range of energy. At this respect, there is an alternative proposal [4] that starts from a modification of the α_s definition that avoids the Landau pole but retains the standard properties. Nevertheless, this change implies a modification in the linear confinement behaviour loosing the standard connection with the string tension.

All the previous mentioned problems can be avoided by using the Generalized Borel Transform (GBT) that was introduced in Ref. [9]. This version of the Borel transform was originally defined on a finite lattice but it can be readily adapted to the continuum, preserving all of its characteristics. The main vantages of this proposal comes from the fact that its analytic properties do not give rise to ambiguities. Moreover, it allows to perform computations in terms of a real and positive arbitrary parameter λ avoiding the implementation of perturbative expansions. The approach generally ends with non-perturbative calculations of the saddle point type, when λ is takes as large as one wants. This is possible because the generalization implies the definition of a valid Borel transformation for each value of the parameter λ and one can choose the one more adapted to own particular problem. In other words, when using the GBT $B_\lambda(s)$

one is performing a whole class of transformations. It is found that, as it should be, the results do not depend on λ. For the particular case of present interest, this can be summarize as

$$a(1/r) = T_\lambda^{-1}\left[T_\lambda\left(a(1/r)\right)\right] \quad where \quad T_\lambda\left(a(1/r)\right) \equiv B_\lambda(s) \quad for \quad 0 < r < \infty$$

We start by presenting our previous generalization [9] of the Borel transform of a function $f(r)$, namely

$$B_\lambda(s) = \int_0^\infty exp[s/\eta(r)] \left[\frac{1}{\lambda\,\eta(r)} + 1\right]^{-\lambda\,s} f(r)\,d(1/\eta(r)) \quad ; \; Re(s) < 0 \tag{8}$$

Among the properties of this definition we want to notice that it is valid for any analytic function $\eta(r)$ in the interval $0 < r < \infty$ that allows to define

$$u_\lambda(r) \equiv \frac{1}{\eta(r)} - \lambda\,\ln\left[\frac{1}{\lambda\,\eta(r)} + 1\right] \tag{9}$$

monotonically increasing in the same interval as soon as $\eta(r)$ is. Consequently, the integral transform can be written as

$$B_\lambda(s) = \int_0^\infty exp[s\,u]\,f[r_\lambda(u)]\,\{1 + \lambda\,\eta[r_\lambda(u)]\}\,du \;\; ; \;\; Re(s) < 0 \tag{10}$$

where $r_\lambda(u)$ is the inverse coming from the change of variables. From the last expression it is clear that

$$B_\lambda(s) = \int_0^\infty exp(s\,u)\,L_\lambda[r(u)]\,du \;\; ; \;\; Re(s) < 0 \tag{11}$$

is the Laplace transform of the function $L_\lambda[r(u)]$ there implicitly defined. That definition implies that $\eta(r)$ gives rise to an analytic transformation in the negative Borel half-plane, such that its extension to the other half-plane is also analytic with a cut on the real positive axis. From this observation it is clear that $f(r)$ can be unambiguously expressed in terms of the inverse Laplace transform integrated on the above mentioned cut (for details see Ref. [9])

$$f(r) = \frac{1}{\lambda\,\eta(r) + 1} \int_0^\infty exp[-s/\eta(r)] \left[\frac{1}{\lambda\,\eta(r)} + 1\right]^{\lambda\,s} \triangle B_\lambda(s)\,ds \tag{12}$$

As it was said before, the parameter λ can take any real positive non zero value generating a continuous family of transformations. A large value of λ could be useful because in this case asymptotic techniques can be used in the calculations.

From this point on, a series of almost trivial calculations follows. Let us only indicate the most important steps. Using the ansatz $1/\eta(r) = \lambda\,[exp(\Lambda\,r/\lambda) - 1]$ which is well defined for $0 < r < \infty$, the function $u_\lambda(r)$ results

$$u_\lambda(r) = \lambda\,[exp(\Lambda\,r/\lambda) - 1] - \Lambda\,r$$

Consequently, one can write

$$B_\lambda(s) = \int_1^\infty dq\, H(q) \int_0^\infty exp(s\,\lambda\, v)\,(v+1)^{-\lambda\,(s+q)} dv \tag{13}$$

with

$$H(q) = \frac{1}{\left[\ln\left(q^2-1\right)\right]^2 + \pi^2}\,\frac{1}{q} \tag{14}$$

and where the change of variable $1/[\lambda\,\eta(r)] = v$ was introduced. Notice that the last integral, for $Re(s) < 0$, represents the confluent hypergeometric function [10] $G(1, -\lambda\, s - \lambda\, q + 2, -s\,\lambda)$. Consequently, in this region $B_\lambda(s)$ is an analytic function and when an analytic continuation to the positive half-plane of s is performed, a cut appears. Introducing the discontinuity of the G function, one gets

$$\triangle B_\lambda(s) = 2\,\pi\,\lambda\, exp(-\lambda\, s) \int_1^\infty dq\, H(q)\, \frac{(\lambda\, s)^{\lambda\,(s+q)-1}}{\Gamma[\lambda(s+q)]} \tag{15}$$

We can now transform back to obtain $f(r)$ from eq.(12)

$$f\,(r) = \frac{1}{\pi}\,\lambda^2\, A_\lambda\,(r) \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \exp\left[G\,(w, t, r, \lambda)\right]\, dw\, dt \tag{16}$$

where

$$A_\lambda\,(r) = 1 - \exp\left[-\Lambda\, r/\lambda\right]$$

and

$$\begin{aligned} G\,(w, t, r, \lambda) \;\; = \;\; & -\lambda\, v\,(t)\; u_\lambda\,(r) + t - \pi\, w - \ln\left[w^2+1\right] - 2\,\ln\left(q\,(w)\right) \\ & - \ln\left\{\Gamma\left[\lambda\left(\lambda\, v\,(t) + q\,(w)\right)\right]\right\} + \lambda\, q\,(w)\;\ln\left[\lambda^2\, v\,(t)\right] - \lambda^2\, v\,(t) \\ & + \left(\lambda^2\, v\,(t) - 1\right)\;\ln\left[\lambda^2\, v\,(t)\right] \end{aligned} \tag{17}$$

with

$$q(w) = \left[1 + e^{-\pi\, w}\right]^{1/2} \quad ; \quad v(t) = e^t$$

The next step is to look for the asymptotic contribution in λ of the double integral in eq.(16). To this end one can use the steepest descent technique in the combined variables (t, w). Consequently one first computes the saddle points $t_0(r)$ and $w_o(r)$ and then one checks the positivity condition [11], in particular when the discriminant $D(t_0, w_0)$ of the second derivatives of G at this point is positive. In so doing one obtains

$$f\,(r) \simeq 2\,\lambda^2\, A_\lambda\,(r)\; \exp\left[G\,(w_o\,(r)\,, t_o\,(r)\,, r, \lambda)\right] \left[\frac{\partial^2 G}{\partial w^2}\,\frac{\partial^2 G}{\partial t^2} \;\; - \left(\frac{\partial^2 G}{\partial w \partial t}\right)^2\right]^{-1/2} \tag{18}$$

the saddle point being

$$t_o = \ln\left[\frac{q_0\,(r)}{F\,(q_0\,(r))}\right] = t_o\,(r) \quad ; \quad w_o = -\frac{1}{\pi}\ln\left[q_0^2\,(r) - 1\right] = w_o\,(r) \tag{19}$$

where $q_0(r)$ is the solution of the implicit equation coming from the extremes of the function G, namely

$$r^2 = \frac{F(q_0)}{\Lambda^2}\left[F(q_0) + \frac{1}{q_0}\right] \tag{20}$$

with

$$F(q_0) = \frac{2}{q_0\,[q_0^2-1]}\left[1 - \frac{2\,q_0^2 \ln(q_0^2-1)}{[\ln(q_0^2-1)]^2 + \pi^2}\right] \tag{21}$$

Notice that $F(q_0)$ should be positive and consequently $q_0 < 2.130156$. On the other hand, from eq.(19), $q_0 > 1$. In fact, moving q_0 between these values, the variable r covers all the positive real axis in a biunivocal way. Moreover, the condition $F(q_0) \neq 0$ implies that $r = 0$ is excluded from the analysis. This is clearly not a drawback of the method.

Going now to the expression (18) of $f(r)$, one finally finds, in the saddle point approximation ($\lambda \to \infty$)

$$f(r) = f(q_0(r)) \cong \frac{e^{-1/2}\sqrt{2}\sqrt{(F(q_0) + 1/q_0)}}{2\sqrt{\pi\, D(q_0)}}\,\frac{1}{[q_0]^{3/2}}\left[q_0^2 - 1\right]\frac{\exp\left[-q_0\,F(q_0)\right]}{\left(\left[\frac{1}{\pi}\,\ln(q_0^2-1)\right]^2 + 1\right)} \tag{22}$$

In Table I we have included the results of our analytic expressions and, for comparison, the corresponding values provided by the numerical integration of expression (5) for different spatial distances in the range of physical interest [12] ($0.1\,fm < r < 1\,fm$) and the deviations found. The analysis corresponds to $\Lambda = 0.4\,GeV$

We have obtained an analytic expression for the quark potential valid for any value of r. This result put in evidence the robustness and generality of the previously introduced generalized Borel transform. Notice that for the particular regions of r previously analyzed (see for example [7]) our results, in the leading order of the saddle point approximation, give rise to absolute values of the uncertainties always much smaller than Λ of QCD ($0.4\,GeV$). At this respect, it is worth mentioning that any perturbative based calculation obtained by means of the standard Borel transform, ends with uncertainties at least of the order of the above mentioned Λ.

Our expressions are clearly useful in any further calculations of physical quantities based on this kind of potentials, particularly the quarkonium spectra and the $Q\bar{Q}$ threshold cross section ([13].

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Table I:

q_0	$r\,(q_0)\ [Gev]^{-1}$	$f\,[Eq.(5)]$	$f\,(q_0)$	δf	$\delta V\ [Gev]$
1.304911	7.0	0.001430	0.001395	0.000034	0.00002
1.379315	5.0	0.004236	0.004192	0.000043	0.00003
1.510868	3.0	0.01380	0.014019	-0.000219	-0.00027
1.832551	1.0	0.05671	0.061246	-0.004536	-0.01689
1.998044	0.5	0.09021	0.097194	-0.006984	-0.05201
2.101136	0.2	0.12960	0.125061	0.004538	0.08448